High-speed video analysis of a cantilever

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Abstract

A cantilever is a beam that is fixed at one end and free to oscillate on the other end, and its motion is described by a fourth-order partial differential equation (i.e. a fourth-order wave equation). In this experiment, a cantilever was set up using a long, flat metal beam, and its motion was captured at 600 fps using high-speed video. The video was analyzed to determine how the frequency of oscillation of the free end of the beam depends on the length of the beam. Results showed that for the case where the beam was pulled down on one end and released from rest, the motion of the free end of the beam was sinusoidal. It was found that the frequency of oscillation was proportional to the inverse of the square of the length of the beam, as predicted by the solution to the wave equation for the beam. Using the curve fit, the rigidity of the beam was found to be $K = 0.0151\ \mathrm{N \cdot m^2}$.

I. INTRODUCTION

A cantilever is a beam with one fixed end and one free end as shown in Figure 1. The vertical displacement $u(x,t)$ of a point at location x on the beam oscillates and is described mathematically by the differential equation in Eq. (1),

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^4 u}{\partial x^4} \tag{1}$$

where $\alpha^2 = K/\rho$, K is the rigidity of the beam in $\mathrm{N \cdot m^2}$, and ρ is the linear density of the beam in kg/m. (Solutions of this differential equation for various boundary conditions are discussed by Wylie[1] and Farlow[2].) This differential equation is called a wave equation for the beam, and it is easier to express the partial derivatives in the wave equation using the notation in Eq. (2).

$$u_{tt} = \alpha^2 u_{xxxx} \tag{2}$$

By separating the variables x and t, the solution $u(x,t)$ is written as a product of a function of x and a function of t, as shown in Eq. (3).

$$u(x,t) = X(x)T(t) \tag{3}$$

The general solutions for each function are:

$$T(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{4}$$

$$X(x) = C\cos(\beta_n x) + D\sin(\beta_n x) + E\cosh(\beta_n x) + F\sinh(\beta_n x) \tag{5}$$

where ω_n are the natural frequencies of the beam, and β_n, A, B, C, D, E, and F are constants that must be determined from the boundary conditions (at $x = 0$ and $x = L$) and the initial conditions (at $t = 0$).

For a beam fixed on one end (at the left) and free to oscillate on the other end (at the right), the boundary conditions are:

$$
\begin{aligned}
\text{at left end} \quad u(0,t) &= 0 \quad \text{left end is fixed at } u = 0 && (6)\\
\text{at left end} \quad u_x(0,t) &= 0 \quad \text{beam is horizontal at left end} && (7)\\
\text{at right end} \quad u_{xx}(L,t) &= 0 \quad \text{bending moment is zero} && (8)\\
\text{at right end} \quad u_{xxx}(0,t) &= 0 \quad \text{shear stress is zero} && (9)
\end{aligned}
$$

The initial conditions refer to the position and velocity of each point x on the beam at $t = 0$. These will be functions that depend on how the beam is initially released. For example, the simplest case is if the free end of the beam is pulled downward and released from rest. In any case,

$$
\begin{aligned}
\text{initial vertical displacement} \quad u(x,0) &= f(x) && (10)\\
\text{initial vertical velocity} \quad u_t(x,0) &= g(x) && (11)\\
& && (12)
\end{aligned}
$$

where $f(x)$ is determined by the shape of the beam at $t = 0$, and $g(x)$ is determined by the velocity profile of the beam at $t = 0$. The beam may be released from rest; however, it is also possible to hit one part of the beam, like the hammer in a piano striking a string, so that $g(x) \neq 0$.

By substituting the solution $u(x,t) = X(x)T(t)$ into the wave equation in Eq. (2), the following quantities can be determined.

1. Natural frequencies: The natural frequencies are:

$$
\omega_n = \sqrt{\frac{K}{\rho L^4}}(\beta_n L)^2 \tag{13}
$$

where the constants $(\beta_n L)$ are the intersections of the curves in the graph in Figure 2. The first three values are $\beta_n L = 1.875$ rad, 4.694 rad, and 7.855 rad, and subsequent values are for $\cos(\beta_n L) = 0$. Note that $\omega_n \propto \frac{1}{L^2}$. Thus, a shorter cantilever will oscillate with a higher frequency. If the cantilever is twice as short, its frequency will be 4 times greater.

2. Coefficients and frequencies for various modes: The formal solution $u(x,t)$ is

$$
u(x,t) = \sum_{n=1}^{\infty} X(x)(Acos\omega_n t + Bsin\omega_n t) \tag{14}
$$

where the constants A and B are determined from the initial conditions.

For the initial vertical displacement,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n X(x) \tag{15}$$

For the initial velocity,

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} (\omega_n B_n) X(x) \tag{16}$$

From the shape of the cantilever at $t = 0$, $f(x)$ can be determined by a curve fit. By measuring the initial velocity of various points on the cantilever, $g(x)$ can be determined by a curve fit. With these known, a Fourier transform can be used to determine the coefficients A_n and B_n for the natural frequencies ω_n. Thus, the Fourier transform tells us which eigenfunctions make up the solution for the given initial conditions.

In this experiment, the free end of the cantilever was released from rest, and it oscillated sinusoidally. High-speed video was used to measure the vertical position of the free end as a function of time. The angular frequency of oscillation was measured by a fitting a sinusoidal function to the graph of vertical position vs. time. The angular frequency was measured as a function of amplitude and as function of the length of the bar.

II. APPARATUS

The apparatus is shown in Figure 3 with the bar flexed and ready to be released. The cantilever was a long, rectangular aluminum bar that was 0.002 m tall, 0.914 meters long and 0.048m wide. Yellow stickers were placed at 10 cm increments on the bar. The right-side of the bar was clamped to a table so that it was fixed. A meterstick with yellow stickers placed 10 cm apart was placed close to the plane of the bar and was used for distance calibration in the video.

III. EXPERIMENT: FREQUENCY AS A FUNCTION OF AMPLITUDE

It is well known that the frequency of a simple harmonic oscillator is independent of the amplitude of oscillation. However, it was not clear if this was the case for the oscillating cantilever beam (free end). To determine whether it was important to control the amplitude of oscillation in the experiment, it was first determined whether amplitude affected the frequency of oscillation.

The bar was clamped at a location so that its length was 50 cm. The free end of the cantilever was pushed downward and released from rest as shown in Figure 3. The vertical position y of the free end of the cantilever was measured as a function of time t for numerous oscillations. The result is shown in Figure 4 along with the a sinusoidal curve fit of the form $y = Asin(Bt + C) + D$. Comparing this with the sinusoidal function $y = Asin(\omega t + \phi)$ shows that the amplitude is the coefficient A and the angular frequency is the coefficient B from the curve fit.

Four different trials were conducted, each with a different amplitude, and the results are shown in Table I. The average frequency is 3.034 rad/s with a standard deviation of 0.009 rad/s. That's a percent deviation of only 0.003% from the mean. Considering that the amplitude was changed by a factor of two, the variation in the angular frequency is statistically insignificant. It is safe to conclude that the amplitude does not affect the angular frequency; therefore, the amplitude does not have to be controlled when measuring the angular frequency of the bar.

IV. EXPERIMENT: FREQUENCY AS A FUNCTION OF THE LENGTH OF THE BAR

According to Eq. (13), the angular frequency of the bar should be proportional to $1/L^2$ of the bar, since $(\beta_n L)$ are constants for different modes of oscillation. To test this, the angular frequency of the free end of the bar was measured for various lengths of the bar: 40 cm, 50 cm, 60 cm, and 70 cm. The amplitude was kept approximately the same, though this was not necessary since amplitude does not affect angular frequency.

For each length, the experiment was repeated for a total of three trials, and the average angular frequency and standard deviation of the angular frequency were measured. The standard deviation of the angular frequency was used to determine the uncertainty in the measurement of the angular frequency. As shown in Figure 4, the angular frequency for each trial was measured using a sinusoidal curve fit of $y(t)$ for the free end of the cantilever.

The results are shown in Table II. Note that the standard deviation is remarkably small, showing that high-speed video analysis in this case is a precise and excellent technique for measuring the frequency of oscillation. The relative error in the worst case is only 0.2%

A graph of ω_{ave} as a function of L is shown in Figure 5. The error bars are too small to be seen on the graph. The function $\omega = A/L^2$ was fit to the data, and the best-fit curve was found to be $\omega = (0.755 \text{ m}^2/\text{s})/L^2$.

It was not established that the free end of the beam oscillates in the fundamental mode ($n = 1$). However, assuming that $n = 1$, then $\beta_1 L = 1.875$ and the curve fit parameter is $A = \sqrt{\frac{K}{\rho}}(1.875)^2 =$

0.755 m^2/s. Solving for the rigidity constant for this particular aluminum bar (linear density $\rho = 0.327$ kg/m) gives $K = 0.0151\ \mathrm{N \cdot m^2}$.

V. CONCLUSION

This experiment investigated the motion of the free end of an oscillating cantilever beam that was displaced on the free end and released from rest. High-speed video analysis was used to measure the amplitude and angular frequency of the free end of the beam. It was found that the angular frequency is independent of the amplitude, which is similar to that of a simple harmonic oscillator. In addition, the angular frequency was found to depend on $1/L^2$ where L is the length of the beam; thus, shorter beams oscillate with a higher frequency, exactly as predicted by the solution of the wave equation. The curve fit constant for the graph of ω as a function of L was used to measure the rigidity of the beam, though it was not proven that the measured oscillation was the fundamental mode with $n = 1$.

In future experiments, it should be possible to show that the measured oscillation is the fundamental ($n = 1$). In addition, the initial conditions could be changed by hammering the center of the beam, for example. In this case, it might be possible to get a superposition of modes. The resulting graph of $y(t)$ would be periodic, but not necessarily sinusoidal. Using Fourier Analysis and Equations (15)and (16), the various modes could be determined.

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[1] C. R. Wylie Jr., *Advanced Engineering Mathematics* (McGraw-Hill, 1966), pp. 323–327.

[2] Stanley J. Farlow, *Partial Differential Equations for Scientists and Engineers* (Dover, 1993), pp. 161-167.

Tables

Amplitude, A (cm)	Angular Frequency, ω (rad/s)
5.826	3.025
7.117	3.030
9.397	3.036
10.980	3.046

TABLE I: Angular frequency for various amplitudes.

Length, L (m)	ω_{ave} (rad/s)	Standard Deviation, σ (rad/s)
0.40	4.705	.00849
0.50	3.024	.00153
0.60	2.107	.00173
0.70	1.555	.00173

TABLE II: Angular frequency for various lengths of the cantiliever.

Figures

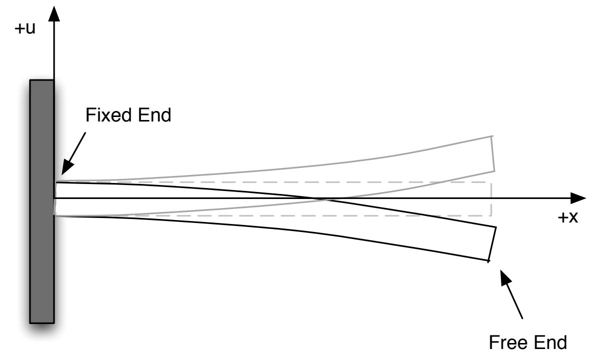

FIG. 1: The cantilever and coordinate system.

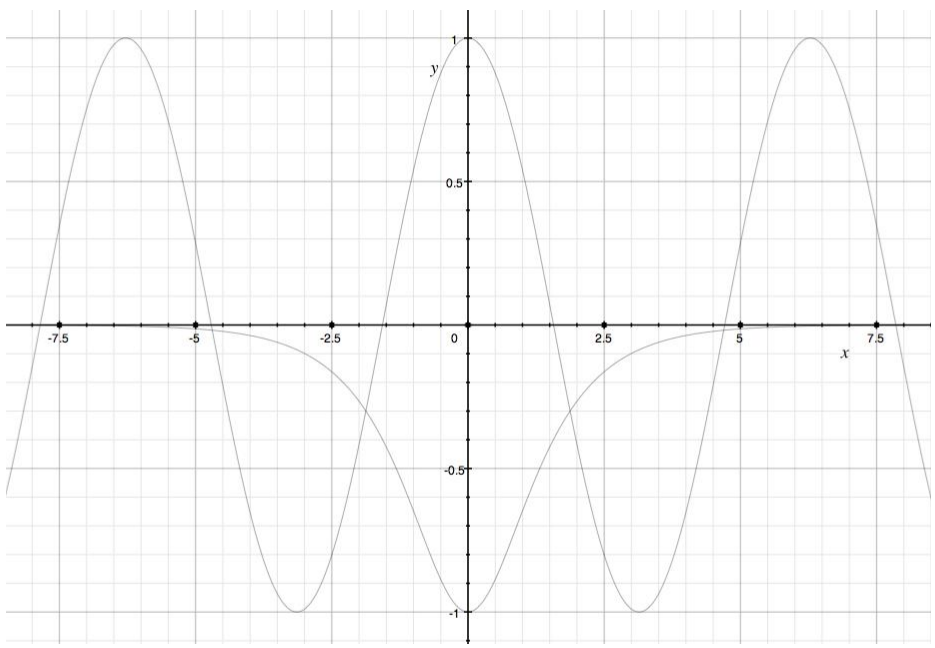

FIG. 2: Graphs of $\cos(\beta_n L)$ and $\frac{-1}{\cosh(\beta_n L)}$.

FIG. 3: A cantilever made of an aluminum bar fixed at one end.

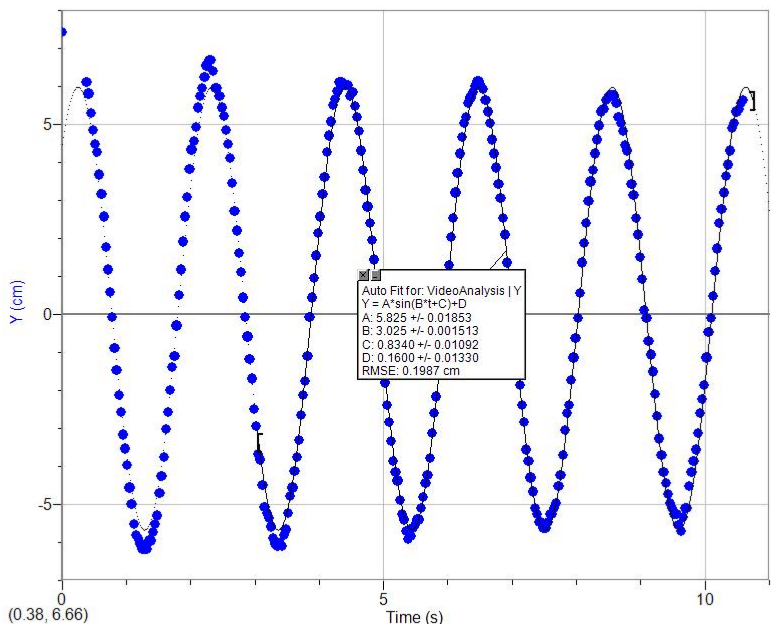

FIG. 4: $y(t)$ for the free end of the cantilever

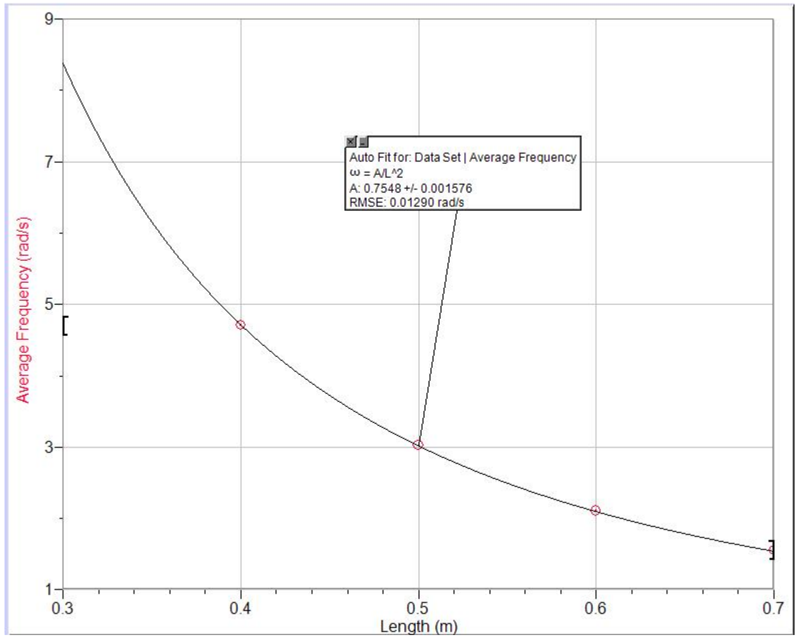

FIG. 5: ω_{ave} vs. L