Circular and Elliptical Orbits

Objective: Define uniform circular motion; define angular speed; describe the net force and velocity vectors for an object in uniform circular motion; apply the momentum principle to uniform circular motion; describe how the *tangential* and *perpendicular* components of the net force vector affect the momentum of the object.

Uniform circular motion

If a planet has just the right speed (for a given radius of orbit), then it will travel in circular motion with a constant speed. This is called *uniform circular motion.* Note that the velocity vector changes direction, although the speed is constant. This means that there must be a net force on the planet.

When writing a simulation of a planet in a circular orbit, we observe:

- The net force on the planet is directed toward the center of the circle and has a constant magnitude.
- The momentum of the planet is tangent to the path and is always perpendicular to the net force. The magnitude of the momentum remains constant.

The speed of the planet is

$$|\vec{v}| = \frac{2\pi r}{T} \qquad \text{only for uniform circular motion} \tag{1}$$

where T is the period of the orbit (the time interval for one complete revolution) and r is the radius of the orbit.

The rate at which something rotates (the change in angular position with respect to time) is called the *angular speed.*

$$\omega = \frac{d\theta}{dt} \qquad \text{definition} \tag{2}$$

For uniform circular motion,

$$\omega = \frac{d\theta}{dt} = \frac{2\pi}{T} \qquad \text{uniform circular motion} \tag{3}$$

In general, the speed can be written as

$$|\vec{v}| = \omega r \qquad \text{general relationship of speed and angular speed} \tag{4}$$

The rate of change of momentum of an object moving in uniform circular motion at low speeds compared to the speed of light is

$$\frac{d\vec{p}}{dt} = -m\omega^2|\vec{r}|\hat{r} \qquad \text{where } \hat{r} \text{ is the unit vector pointing outward from the center of the circle} \tag{5}$$

showing that the net force on the object points toward the center of the circle and has a constant magnitude.

Circular Gravitational Orbit

A circular orbit of a satellite around a planet is an example of uniform circular motion (its speed remains constant). Thus, the momentum principle applied to the satellite gives us

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = -m\omega^2 r\hat{r} \tag{6}$$

where $\hat{r}$ is the unit vector pointing outward from the center of the circle. The net force, however, is equal to the gravitational force of the planet on the satellite; therefore,

$$\vec{F}_{\text{net}} = \vec{F}_{\text{by earth on sat}} = -\frac{GMm}{r^2}\hat{r} \tag{7}$$

where $\hat{r}$ is the unit vector pointing outward from the center of the circle. NOTE: this is a different direction than we defined $\hat{r}$ in our VPython program, but that's ok because we put a negative sign out front. Equating these equations for net force gives us

$$-\frac{GMm}{r^2}\hat{r} = -m\omega^2 r\hat{r} = -m\frac{v^2}{r}\hat{r} \tag{8}$$

Thus, the speed of a satellite in a circular orbit at radius r is

$$v = \sqrt{\frac{GM}{r}} \tag{9}$$

If you want to know the period, use

$$v = \frac{2\pi r}{T} \qquad \text{only for uniform circular motion} \tag{10}$$

Elliptical orbits

You will observe the following characteristics of an elliptical orbit of a planet around a star:

- The net force on the planet is directed toward the star. Its magnitude changes depending on the proximity of the planet to the star. The gravitational force is larger near the star and smaller far from the star.
- The momentum of the planet is tangent to the path. The planet travels fastest nearest the star and slowest furthest from the star.
- When the planet is slowing down, the component of the net force on the planet that is tangent to the path is opposite to the momentum of the planet (i.e. *anti-parallel*).
- When the planet is speeding up, the component of the net force on the planet that is tangent to the path is in the same direction as the momentum of the planet (i.e. *parallel*).

As opposed to uniform circular motion, the momentum of the planet changes in both magnitude and direction. It turns out that the change in magnitude of the momentum is related to the tangential component of the net force on the planet, and the change in direction of the momentum is related to the perpendicular component of the net force on the planet. See Figure 1

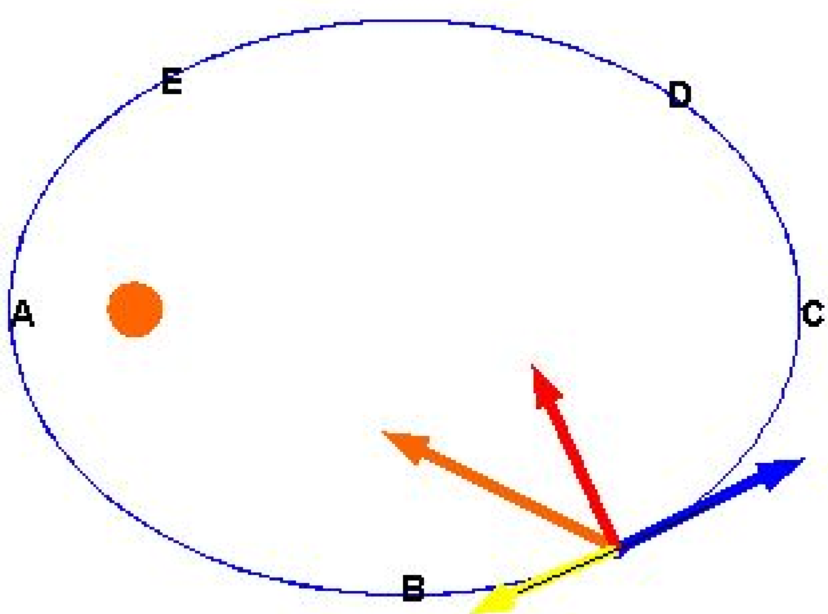

Figure 1: The momentum and net force vectors (along with the tangential and perpendicular components of the net force) are shown for a planet orbiting a star. Can you identify and label the vectors?

Application

1. Suppose satellite A orbits at twice the orbital radius as satellite B. Which has the greater period and by how much?

2. If a satellite orbiting the earth with constant speed should orbit with a period of 24 hours, what should its orbital radius be?

3. Label the vectors in Figure 1.

4. If the planet is in a circular orbit, what would the vectors look like? Make a sketch.

5. A merry-go-round takes 10 seconds to go around once. What is the angular speed? What are the units of the angular speed?

6. If the radius of this merry-go-round is 8 meters, what is the speed at the outer rim? What is the direction of the velocity?

7. In outer space, a ball of mass 5 kg at the end of a spring moves in a circle of radius 2 m at a constant speed of 3 m/s. What is the force exerted by the spring on the rock?

8. Suppose you ride a Ferris wheel that rotates with constant speed.

   (a) Write an expression for the net force on you at any point on the ride? What can you say about its direction at any point on the ride?

   (b) At which point will the magnitude of the net force of the chair on you be the greatest?

   (c) We know that the earth always pulls downward with a force approximately equal to mg. If the only other object you interact with is the chair (including seat belt, etc.), what must be the direction of the force of the chair on you at the top? at the bottom? at the right of center? at the left of center?